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Geometric quantization and constraints in field theory*

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Abstract. *The main objective of this series of lectures is a discussion of the problem of quantization of systems with constraints, first studied by P.A.M. Dirac. I want to reinterprete Dirac's approach to quantization of constraints in the framework of geometric quantization, and then use it to discuss some aspects of quantized Yang--Mills fields.*
We begin with a review of geometric quantization and the implied relationship between the co-adjoint orbits and the irreducible unitary representations of Lie groups. Next, we discuss an intrinsic Hamiltonian formulation of a class of field theories which includes gauge theories and general relativity. Quantization of this class of field theories is discussed. Dirac's approach to quantization of constraints is reinterpreted in the framework of geometric quantization.

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Chapter 1

ORBITS AND REPRESENTATIONS

1. MOMENTUM MAP

Let (P, ω) be a symplectic manifold. That is P is a smooth manifold and ω is a closed non-degenerate 2-form on P. For each function f on P, we denote by ξ_f the Hamiltonian vector field of f defined by

$$\xi_f \lrcorner\, \omega = -\mathrm{d}f,$$

where $\lrcorner$ denotes the left interior product of forms by vector fields,

$$\langle \xi \lrcorner\, \omega, \eta \rangle = \langle \omega, \xi \wedge \eta \rangle.$$

The map $f \to \xi_f$ induces the structure of a Lie algebra in the space of smooth functions on P, called the Poisson algebra of (P, ω). The Poisson bracket of f and h is given by

$$[f, h] = -\langle \omega, \xi_f \wedge \xi_h \rangle = -\xi_f h = \xi_h f.$$

We say that an action of a connected Lie group G in P is Hamiltonian if there exists a G equivariant mapping $J : P \to g^*$, where g^* is the dual of the Lie algebra g of G, such that, for each ξ in g, this action of $\exp(-t\xi)$ in P is given by the translation by t along the integral curves of the Hamiltonian vector field of the function J_ξ on P defined by

$$J_\xi(p) = \langle J(p), \xi \rangle$$

for every p in P. The map J is called a momentum map for the action of G in (P, ω), and the function J_ξ is called a momentum associated to ξ.

Examples

1. Consider $P = \mathbf{R}^6$ with the canonical coordinates (p_i, q^j), $i, j = 1, 2, 3$, $\omega = \Sigma_i \, \mathrm{d}p_i \wedge \mathrm{d}q^i$, and $G = SO(3)$. The action of G in P is given by the fundamental action of $SO(3)$ on the variables q^i, and its transpose on the variables p_j. The Lie algebra $so(3)$ is 3-dimensional, and a momentum map gives the angular mementum.

2. Let $P = \mathbf{R}^{2n}$ with the canonical coordinates (p_i, q^j), $i, j = 1, \ldots, n$, the symplectic form as before, and $G = Sp(n, \mathbf{R})$, the group of all linear transforma-

tions preserving ω. Components of a momentum mapping are quadratic homogeneous polynomials on P.

3. Let G be a connected Lie group, and P an orbit of the co-adjoint action of G in g^*. For each ξ in g, let ξ_P denote the fundamental vector field on P corresponding to ξ, that is the translation by t along the integral curves of ξ_P gives the action of $\exp(t\xi)$ in P. Since P is diffeomorphic to G/G_p, where p is any point in P, and G_p is the stability group of p, the tangent bundle space of P is globally spanned by the fundamental vector fields. The canonical symplectic form of the orbit P is the unique form ω such that, for every p in P, and every ξ and η in g,

$$\langle \omega, \xi_P(p) \wedge \eta_P(p) \rangle = \langle p, [\xi, \eta] \rangle.$$

The momentum mapping for the action of G on P induced by the co-adjoint action in g^* is given by the embedding $J : P \to g^*$. For each ξ in g, the momentum $J_\xi : P \to \mathbf{R}$ is given by $J_\xi(p) = \langle p, \xi \rangle$, [18].

In the first example, a quantization of (P, ω) gives rise to a representation of $SU(2)$, that is the double covering group of $SO(3)$. Similarly, in the second example, the Schroedinger quantization gives rise to a representation of the double covering $Mp(n, \mathbf{R})$ of the symplectic group; the metaplectic representation. In the third case, the orbit P is a general symplectic manifold, and one needs a quantization scheme applicable to this case. Such a quantization scheme, called geometric quantization, was introduced independently by B. Kostant, [18], [19], and J.-M. Souriau [25].

2. GEOMETRIC QUANTIZATION

Given a G orbit P in g^* with the canonical symplectic form ω, let L be a complex line bundle over P with a connection ∇ satisfying the prequantization condition

$$\text{curvature } \nabla = -h^{-1} p_{PL}{}^* \omega$$

where h is the Planck's constant, and $p_{PL} : L \to P$ is the line bundle projection. [The Planck's constant is introduced here in order to relate geometric quantization to quantum mechanics. These results in rescaling some of the formulae of the geometric quantization]. A line bundle satisfying the prequantization condition exists if and only if $h^{-1}\omega$ defines an integral de Rham cohomology class and, if this condition is satisfied, the set of equivalence classes of such line bundles with connections can be parametrized by the group of all unitary characters of the fundamental group of P.

For each function f on P, we denote by Pf the differential operator on the space of sections of p_{PL} defined by

$$Pf = -i\hbar\nabla_{\xi_f} + f,$$

where $\hbar$ is the Planck's constant divided by 2π, and ξ_f is the Hamiltonian vector field of f. The assumed relationship between the curvature of ∇ and the symplectic form ω implies that, for every pair f_1 and f_2 of functions on P,

$$[Pf_1, Pf_2] = i\hbar P[f_1, f_2].$$

A momentum mapping J gives rise to a homomorphism of the Lie algebra g of G into the Poisson algebra of (P, ω). Associating to each ξ in g the differential operator $-i\hbar^{-1}PJ_\xi$ we obtain a linear representation of the Lie algebra g of G in the space of sections of L, [19]. The representation of G obtained by integrating this representation of g is called the prequantization representation corresponding to (L, ∇). The prequantization representation is not irreducible, and we need to restrict the representation space to a subspace carrying an irreducible representation of G.

In order to reduce the prequantization representation one introduces a polarization of (P, ω), that is an involutive complex Lagrangian distribution F on P such that,

$$D = F \cap \overline{F} \cap TP, \quad \text{and} \quad E = (F + \overline{F}) \cap TP,$$

where $\overline{F}$ denotes the complex conjugate of F, are involutive distributions on P, and the spaces P/D and P/E of integral manifolds of D and E, respectively, are quotient manifolds of P. If F is invariant under the action of G, then the prequantization representation restricted to the space of sections of L, which are covariantly constant along F, is in many cases irreducible.

Unless $E = TP$, there is no natural scalar product in the space of sections of L, which are covariantly constant along F, and the obtained irreducible representation of G is not unitary. In order to unitarize this representation, one introduces a metaplectic structure, that is a double covering space of the bundle of symplectic frames in TP, which is a right principal $Mp(n, \mathbf{R})$ bundle over P, where $n = (1/2) \dim P$. For each polarization F, the metaplectic structure gives rise to a complex line bundle $\sqrt{\Lambda^n F}$ over P endowed with a partial connection covering F. Moreover, there is a sesquilinear mapping $\langle , \rangle$ from the space of sections of $\sqrt{\Lambda^n F}$ to the space of densities on P/D. For a G invariant polarization F the space H_F of sections of $L \otimes \sqrt{\Lambda^n F}$, which are covariantly constant along F and push down to square integrable densities on P/D, carry an irreducible unitary representation of G.

The geometric quantization construction yields all irreducible unitary representations of nilpotent Lie groups, [16], and of solvable Lie groups of type I, [2]. For semisimple Lie groups not all representations can be obatained in this way.

Let F and F' be two different polarizations and H_F and $H_{F'}$ the corresponding representation spaces. The relationship between H_F and $H_{F'}$, can be studied in terms of a sesquilinear mapping $K : H_F \times H_{F'} \to \mathbf{C}$, introduced by R.J. Blattner, B. Kostant, and S. Sternberg.

Blattner-Kostant-Sternberg kernels can be also used to study co-adjoint orbits which do not admit G invariant polarizations, [5], [6].

The geometric quantization can be also applied to symplectic manifolds which are not co-adjoint orbits but phase spaces of dynamical systems. For most physically interesting systems with finite numbers of degrees of freedom, the geometric quantization construction leads from the classical phase space of the system to the corresponding quantum theory, [21]. One exception is the Dirac theory of electron in an external electromagnetic field, for which we cannot find a classical phase space which would yield the correct quantum theory.

3. ORBITS AND REPRESENTATIONS

Geometric quantization associates to some co-adjoint orbits the corresponding irreducible unitary representations of connected Lie groups. Conversely, to each unitary irreducible representation ρ of a connected Lie group G, one can associate a set of co-adjoint orbits as follows.

The infinitesimal character of ρ is a homomorphism λ_ρ from the centre $Z(g)$ of the universal enveloping algebra $U(g)$ into the field $\mathbf{C}$ of complex numbers. The linear space $Z(g)$ can be identified with the space of G invariant polynomials on g^*. Invariant polynomials are constant on G-orbits in g^*. A G orbit P corresponds to the representation ρ if, for every invariant polynomial P on g^*,

$$\lambda_\rho(P) = P(-h^{-1}p),$$

where p is any point in P, and the factor $-h^{-1}$ appears because we have introduced the Planck's constant in the prequantization condition in Section 2.

Let G be a connected and simply connected nilpotent Lie group, and H a connected closed subgroup of G. The geometric quantization yields a bijection from the sets of co-adjoint orbits of G and H to the equivalence classes of irredicible unitary representations of G and H, respectively. Moreover, if an irreducible unitary representation ρ of G corresponds to an orbit P of G in g^*, then its restriction to H is decomposed into the direct integral of irreducible unitary representations of H which correspond to the orbits of H contained in $p_{g^*h^*}(P)$, where $p_{h^*g^*} : g^* \to h^*$ is the canonical projection induced by the embedding of

h into g, [16]. It should be noted that the projection $p_{h^*g^*}$ restricted to a G orbit P in g^* induces the momentum mapping $J : P \to h^*$ corresponding to the Hamiltonian action of H in P. Thus, if P_H is an H orbit in h^*, then the representation of H corresponding to the orbit P_H appears in the decomposition of the restriction of ρ to H if and only $J^{-1}(P_H)$ is not empty.

The results above suggest the following problem. Let (P, ω) be a symplectic manifold with a Hamiltonian action of a connected Lie group H, and let $J : P \to h^*$ be the corresponding momentum map. Suppose we have a quantization structure in (P, ω) leading to a representation ρ of H. We want to relate the decomposition of ρ into irreducible representations of H to the partition of P by the inverse images $J^{-1}(P_H)$ of co-adjoint orbits P_H of H under the momentum mapping J.

If J is transverse to P_H, then $J^{-1}(P_H)$ is a submanifold of P. Moreover, if the space $J^{-1}(P_H)/H$ of the H orbits in $J^{-1}(P_H)$ is a quotient manifold of $J^{-1}(P_H)$, then it has a canonically defined symplectic structure. If the quantization structure in (P, ω) induces a quantization structure in $J^{-1}(P_H)/H$, then one could expect that the quantization of $J^{-1}(P_H)/H$ will yield some information about the appearance in the decomposition of ρ of the representations of H corresponding to the orbit P_H.

In particular, 0 in h^* is the H orbit corresponding to the trivial representation of H, and $J^{-1}(0)/H$ is the reduced phase space of the zero level of the momentum map J, [15]. Hence the quantization of the reduced phase space $J^{-1}(0)/H$ should yield information about the appearance of the trivial representation in the decomposition of the representation ρ. If P and H are compact, and (P, ω) admits a prequantization line bundle and a positive complex polarisation, then the reduced phase space $J^{-1}(0)/H$ inherits a quantization structure, and the representation space obtained by quantization of $J^{-1}(0)/H$ is canonically isomorphic to the space of H invariant vectors in the representation space obtained by the quantization of (P, ω), i.e. the trivial component of ρ, [13]. Under appropriate assumptions a similar result can be obtained for non-compact P and H and a real polarization, [23].

The results stated above require that 0 should be a regular value for the momentum map J. If 0 is not a regular value of J, one can construct a Poisson algebra, which in the regular case corresponds to the Poisson algebra of the reduced phase space. Extending the ideas of geometric quantization to Poisson algebras one can hope to recover the correspondence between the trivial component of ρ and the quantization of the reduced Poisson algebra; such a correspondence can be shown in an example, [24].

Chapter 2

HAMILTONIAN DYNAMICS OF CLASSICAL FIELDS

Our aim is to apply to field theory the insight about the relationship between the co-adjoint orbits and the irreducible unitary representations of Lie groups, which we have learned from geometric quantization. Since geometric quantization has been designed for symplectic manifolds, I have to discuss first the canonical formulation of field theory; that is the geometric structure behind the adjoint formalism used by AMM, (J. Arms, J. Marsden and V. Moncrief, see [1] and the references quoted there) following GIMMSY (M. Gotay, J. Isenberg, J. Marsden, R. Montgommery, J. Sniatycki, and Ph. Yasskin, [12]).

In the formulation proposed here the group G of symmetries of the theory plays a fundamental role. All the structures introduced in the theory have to be invariant under G. If, in order to achieve the required formulation of the theory, we have to introduce an external object not preserved by the symmetry group G, we introduce a G orbit of such external objects, so that the resulting theory remains G invariant. For example, it is well known that, in order to obtain the canonical equations of motion for relativistic classical fields, we have to treat them as equations of evolution for the Cauchy data on some Cauchy surface. However, a Cauchy surface is not invariant under the Poincarè group G. Hence, in order to recover the relativistic invariance of the resulting theory we have to take a G orbit of Cauchy surfaces.

4. DE DONDER-WEYL THEORY

The DeDonder-Weyl theory of the calculus of variations, [9] and [26], appears to be a useful tool in a realization of the program outlined above. In order to establish the notation I shall give a brief review of the essential points of DeDonder-Weyl theory following [20].

Let X be an oriented 4-dimensional manifold representing the space-time, $p_{XY}: Y \to X$ a locally trivial fibre bundle, Z the space of 1-jets of sections of p_{XY}, $p_{XZ}: Z \to X$ the source map, and $p_{YZ}: Z \to Y$ the target map. For each $x \in X$, the fibre $Z_x = p_{XZ}{}^{-1}(x)$ consists of equivalence classes of sections s_{YX} of p_{XY} under the equivalence relation $\sim_x$ given by

$$s_{YX} \sim_x s'_{YX} \iff T_x s_{YX} = T_x s'_{YX},$$

where $T_x s_{YX}$ denotes the restriction of Ts_{YX} to $T_x X$. The $\sim_x$ equivalence class of a section s_{YX} is denoted by $j_x s_{YX}$. The source and the target maps are given by

$$p_{XZ}(j_x s_{YX}) = x, \quad \text{and} \quad p_{YZ}(j_x s_{YX}) = s_{YX}(x),$$

respectively. The fibres of the target map p_{YZ} are affine spaces.

Each section s_{YX} of p_{XY} given rise to a section of p_{XZ}, called the jet extension of s_{YX} and denoted js_{YX}, defined by

$$js_{YX}(x) = j_x s_{YX},$$

for every $x \in X$. Clearly, $p_{YZ} = js_{YX} = s_{YZ}$.

The canonical form of the first jet space is the mapping $D : TZ \to \text{Ker}\ Tp_{XY}$ defined as follows. Given $z = J_X s_{YX}$ in Z and a vector $u \in T_z Z$,

$$D(u) = Tp_{YZ}(u) - Ts_{YX}(Tp_{XZ}(u)).$$

A section s_{ZX} of the source map p_{XZ} is the jet extension of its projection to Y, $s_{ZX} = j(p_{YZ} s_{ZX})$, if and only if

$$DTs_{ZX} = 0.$$

Let ξ_Y be a vector field on Y projecting to a vector field ξ_X in X, then there exists a unique vector field ${}_Z$ on Z, projecting to ξ_Y and preserving the canonical form D. We shall refer to ξ_Z as the canonical extension of ξ_Y to Z.

A Lagrangian is a 4-form Λ on Z such that $u \lrcorner \Lambda = 0$ whenever $u \in \text{Ker}\ Tp_{XZ}$. A local section s_{YX} of p_{XY}, with a relatively compact domain U, is a stationary point of the action integral corresponding to a Lagrangian Λ if, for each vector field ξ_Y on Y vanishing on the boundary of U,

$$\int js_{YX}{}^* \pounds_{\xi_Z} \Lambda = 0,$$

where ξ_Z is the canonical extension of ξ_Y to Z. Stationary points of the action integral are solutions of the Euler-Lagrange equations corresponding to the Lagrangian Λ.

Since $DTjs_{YX} = 0$, for every section s_{YX} of p_{XY}, it follows that a modification of Λ off $\Lambda^4 \text{Ker}\, D$ does not change the action integral. There exists a unique 4-form Ω satisfying the following conditions:

(i) $$u \in \text{Ker}\ Tp_{YZ} \Rightarrow u \lrcorner \Omega = 0,$$

(ii) $$u_1, \ldots, u_4 \in \text{Ker}\, D \Rightarrow \langle \Omega - \Lambda, u_1 \wedge \ldots \wedge u_4 \rangle = 0,$$

(iii) $u_1, \ldots, u_5 \in \operatorname{Ker} D \Rightarrow \langle d\Omega, u_1 \wedge \ldots \wedge u_5 \rangle = 0,$

(iv) $u_1, u_2 \in \operatorname{Ker} Tp_{XZ} \Rightarrow \langle u_1 \wedge u_2 \rangle \lrcorner \Omega = 0.$

The form Ω defined here was called by Goldschmidt and Sternberg, [10], the Hamilton-Cartan form corresponding to the Lagrangian Λ. However, E. Cartan, [7], atributes it to Th. DeDonder, and I shall refer to Ω as the DeDonder form corresponding to Λ. The variational principle with replaced by Ω leads to the DeDonder-Weyl theory in the calculus of variations. Substituting the form Ω into the action integral, and taking into account the conditions (i) through (iv) and the fundamental theorem in the calculus of variations one concludes that a section s_{YX} is a stationary point of the action integral if and only if

$$js_{YX}{}^*(\xi \lrcorner d\Omega) = 0,$$

for every vector field ξ on Z.

For Lagrangian quadratic in derivatives [this notion makes sense since the fibres of the target map p_{YZ} are affine spaces] one can show that a section s_{YX} of p_{XY} satisfies the Euler-Lagrange equations if and only if it is a projection to Y of a section s_{ZX}, $s_{YX} = p_{YZ} s_{ZX}$, such that

$$s_{ZX}{}^*(\xi \lrcorner d\Omega) = 0,$$

for every vector field ξ on Z, [22]. I shal refer to the above equation as the De Donder-Weyl equation.

5. Symmetries

There are various notions of symmetries of field theory, depending on which part of the structute of the theory is to be preserved under a symmetry transformation. Here, we use the most restrictive notion of symmetry, requiring that a symmetry should preserve all the structure of the Lagrangian theory of the system under consideration. In physical applications most symmetries of importance are of this type, in particular the symmetries related to the relativistic invariance and the gauge invariance in Yang-Mills theory and general relativisty satisfy this condition.

Thus, by a symmetry of the theory we understand a diffeomorphism g_Y of Y inducing a diffeomorphism g_X of X such that $g_X p_{XY} = p_{XY} g_Y$, and such that the induced diffeomorphism g_Z of Z preserves the Lagrangian,

$$g_Z{}^*\Lambda = \Lambda.$$

Since the DeDonder form Ω correspondig to Λ, is defined in terms Λ and the structure of the jet bundle, it follows that symmetries preserve Ω. Similarly, an infinitesimal symmetry is a vector field ξ_Y on Y projecting to a vector field ξ_X on X, and such that its extension ξ_Z to Z preserves Λ,

$$\pounds_{\xi_Z} \Lambda = 0.$$

Clearly, ξ_Z preserves also Ω.

We denote by G the group of symmetries of the theory under consideration and by g its Lie algebra. Elements of g are infinitesimal symmetries, however not all infinitesimal symmetries need to integrate to 1-parameter subgroups of G. Thus, the Lie algebra g of G is a Lie subalgebra of the Lie algebra of infinitesimal symmetries. Since elements of G are diffeomorphisms of Y, we denote them by g_Y to distinguish them from the induced diffeomorphisms of other spaces. However, we can consider G as a abstract group acting in various spaces. In this case elements of G will be denoted by g, and g_X, g_Y, and g_Z will denote the action of g in X, Y, and Z, respectively. Similarly, if there is no confusion possible, elements of g, treated as an abstract Lie algebra, will be denoted by single letters, e.g. ξ, and ξ_X, ξ_Y, and ξ_Z will denote the corresponding fundamental vector fields in X, Y, and Z, respectively.

The relationship between infinitesimal symmetries and conservation laws is given by the First Noether Theorem. Let U be a relativaly compact domain in X with boundary expressible as a difference of two hypersurfaces s_1 and s_2,

$$U = s_2 - s_1 .$$

If, ξ is an infinitesimal symmetry then,

$$\xi_Z \lrcorner \, d\Omega = -d(\xi_Z \lrcorner \, \Omega)$$

and, for each section $s_{ZX} : U \to Z$ of p_{XZ} satisfying the DeDonder-Cartan equations, we have

$$\int_{s_1} s_{ZX}{}^*(\xi_Z \lrcorner \, \Omega) = \int_{s_2} s_{ZX}{}^*(\xi_Z \lrcorner \, \Omega).$$

A symmetry $g \in G$ will be called a gauge symmetry if, for every pair of open sets U and U' in X with disjoint closures, there exists a symmetry $g' \in G$ such that

$$g'(y) = g(y) \qquad \text{for every} \qquad y \in p_{XY}{}^{-1}(U),$$

$$g'(y) = y \qquad \text{for every} \qquad y \in p_{XY}{}^{-1}(U').$$

The gauge symmetries form a normal subgroup of G, which we denote by H. The Lie algebra of H will be denoted by h.

The notion of gauge symmetries given above agrees with the commonly accepted notion of gauge symmetries in gauge theories. In general relativity, the group of diffeomorphisms of the space-time manifold satisfies our conditions for gauge symmetries.

Let U be an open set in X bounded by two surfaces s_1 and s_2. The Second Noether Theorem implies that, for each section s_{ZX} with domain U satisfying the DeDonder-Cartan equations, the constants of motion corresponding to all infinitesimal gauge symmetries, such that the intersections of their supports with s_1 and s_2 can be separated by open sets, have to vanish. This consequence of the Second Noether Theorem is the main source of constraints in gauge field theories and general relativity.

6. PRIMARY CONSTRAINTS

Let M be an oriented 3-dimensional manifold representing a typical Cauchy surface. A parametrized Cauchy surface in X is an embedding s of M into X. Let S be a submanifold of the space of embeddings of M into X such that $g_X s \in S$ for each $s \in S$ and each $g \in G$. It is the space of parametrized Cauchy surfaces we are going to admit in our canonical formulation of field dynamics.

Let Q denote the space of embeddings $q : M \to Y$ such that $p_{XY}q$ is in S. An embedding q in Q represents Dirichlet data on the parametrized Cauchy surface $s = p_{XY}q$, that is q describes the values of the dynamical fields on the Cauchy surface $s(M)$. Q is a submanifold of the manifold of smooth maps from M to Y. It is fibered over S with the projection map $P_{SQ} : Q \to S$ given by

$$p_{SQ}(q) = p_{XY}q$$

for every q in Q. The space Q is the field theory analogue of the configuration space-time of the Newtonian dynamics.

Let V denote the space of embeddings $v : M \to Z$, such that $p_{YZ}v$ is contained in Q and there exists a local section s_{YX} of p_{XY} such that

$$v = js_{YX}p_{XZ}v.$$

The space V is the space of «virtual» Cauchy data for the DeDonder-Cartan equations of the field theory under consideration. In the presence of constraints the actual Cauchy data, that is the Cauchy data of solutions of the Hamilton-Cartan equations form a proper subset C of V. We denote by p_{QV} the projection from V to Q given by $p_{QV}(v) = p_{YZ}v$, and by $p_{SV} : V \to S$ the composition of p_{QV} and p_{SQ}, $p_{SV} = p_{SQ}p_{QV}$. For each s in S, the fibre V_s of V over s is the space of «virtual» Cauchy data on the Cauchy surface $s(M)$.

The Legendre transformation is a mapping $L : V \to (\text{Ker } Tp_{SQ})^*$ defined as follows. For each $q \in Q$, each $v \in V_q = P_{QV}{}^{-1}(q)$, and each vector $\xi \in \mathrm{Ker}_q\, Tp_{SQ}$,

$$\langle L(v), \xi \rangle = \int v^*(\xi' \lrcorner\, \Omega),$$

where ξ' is the vector field on Z, defined on $v(M)$, such that $\xi = \xi' v$. The image P of the Legendre transformation L,

$$P = L(V),$$

is the primary constraint bundle of the theory. For each $s \in S$, $P_s = L(V_s)$ is the primary constraint manifold corresponding to the Cauchy surface s.

In sufficiently regular field theories, e.g. general relativity and Yang-Mills theory, the projection map p_{PV} from V to P induced by L is a submersion, the cotangent bundle projection $T^*Q \to Q$ induces a fibration $P_{QP} : P \to Q$, and there exists a unique 1-form θ on P such that, for every $v \in V$ and every $\xi \in T_v V$,

$$\langle p_{PV}{}^*\theta, \xi \rangle = \int v^*(\xi' \lrcorner\, \Omega),$$

where ξ' is the vector field on Z, defined on $v(M)$, such that $\xi = \xi' v$. For each $s \in S$, the fibre of $(\mathrm{Ker}\; Tp_{SQ})^*$ over s is canonically isomorphic to T^*Q_s, where $Q_s = p_{SQ}{}^{-1}(s)$. The pull back θ_s of θ to P_s coincides with the pull back to P_s of the canonical 1-form of the cotangent bundle T^*Q_s by the inclusion map $P_s \to T^*Q_s$.

For each open set U in X, we denote by U_S the subset of S consisting of all $s \in S$ such that $s(M)$ is contained in U. Each local section s_{YX} of p_{XY} with domain U gives rise to a section s_{PS} of $p_{SP} = p_{SQ} p_{QP}$ with domain U_S such that, for each $s \in U_S$,

$$s_{PS}(s) = L(j s_{YX} s).$$

Suppose U_S is open and non-empty. Then a section s_{YX} over U satisfies the DeDonder-Cartan equations if and only if, for each vector field ξ on P,

$$s_{PS}{}^*(\xi \lrcorner\, d\theta) = 0,$$

Thus, we have arrived at a formulation of dynamics of classical fields analogous to the Hamilton-Cartan formulation of time dependent dynamics.

In there are no constraints in the theory and all the infinite dimensional manifolds are suitably chosen, then $P = (\mathrm{Ker}\; Tp_{SQ})^*$, and

$$\mathrm{Ker}\, d\theta = \{\xi \in TP \mid \xi \lrcorner\, d\theta = 0\}$$

is an integrable distribution on P transverse to the fibres of p_{SP}. In the presence of constraints

$$N = \text{Ker}\, d\theta \cap \text{Ker}\, Tp_{SP}$$

does not vanish. For sufficiently regular theories, e.g. general relativity and Yang--Mills theory, N is an integrable distribution on P and the space R of integral manifolds of N is a quotient manifold of P. There is a unique closed 2-form ω on R such that

$$d\theta = p_{RP}{}^*\omega,$$

where $p_{RP}: P \to R$ is the canonical projection. R is the reduced phase bundle of the primary constraint bundle P. It is fibred over S by a map p_{SR} such that $p_{SR}\, p_{RP} = p_{SP}$. For each $s \in S$, the pull back ω_s of ω to the fibre R_s is a symplectic form. (R_s, ω_s) is the reduced phase space corresponding to the Cauchy surface s.

For a physically important class of field theoreis, which includes general relativity and Yang-Mills theory, there is a canonically defined bundle A over S, with the projection map p_{SA}, such that the primary constraint bundle has a canonical product structure over S (Whitney sum),

$$P = R \times_S A.$$

This product structure is stable under the action of the group G of symmetries of the theory. A is called the atlas bundle of the theory. For each $s \in S$, elements of $A_s = p_{SA}{}^{-1}(s)$ are called atlas fields on the Cauchy surface s. The subgroup H_s of the gauge group H stabilising s acts transitively on $A_s = p_{SA}{}^{-1}(s)$.

We denote by $p_{AP}: P \to A$ the projection defined by the product structure. For each $a \in A$, the fibre $P_a = p_{AP}{}^{-1}(a)$ is diffeomorphic to R_s, where $s = p_{SA}(a)$, and it inherits from R_s its symplectic form ω_s. Thus, P has a structure of a symplectic fibration. The closed form $d\theta$, estending the family of symplectic forms on the fibres, defines a distribution hor TP in TP transverse to p_{AP} defined by

$$\text{hor}\, TP = \{\xi \in TP \mid \eta \in \text{Ker}\, Tp_{AP} \Rightarrow \langle \xi \lrcorner d\theta, \eta\rangle = 0\}.$$

It is a connection on P considered as a fibre bundle over A, [11].

7. HAMILTONIAN DYNAMICS

Let $c_S: \mathbf{R} \to S$ be a curve in S. For each $t \in \mathbf{R}$ and each m M, we denote by $\langle c_S(t), m\rangle$ the point in X associated to m by the embedding $c_S(t)$. The curve c_S defines a 3 + 1 splitting of the space-time X if the mapping from $\mathbf{R} \times M$ to X, associating to each (t, m) the point $\langle c_S(t), m\rangle$ is a diffeomorphism. If c_Q is a curve in Q such that $c_S = p_{SQ} c_Q$ defines a $3 + 1$ splitting of the space time manifold X, then it defines a section s_{YX} of p_{XY} such taht $\langle c_Q(t), m\rangle = s_{YX}(\langle c_S(t), m\rangle)$. We want to determine the conditions for curves c_P in P which guarantee that the section s_{YX} determines by $c_Q = p_{QP} c_P$ satisfies the Euler-Lagrange equations provided $c_S = p_{SP} c_P$ defines a $3 + 1$ splitting of the space-time.

The construction given in the preceding section is invariant under the symmetry group G of the theory. Hence, G acts in S, Q, V, P, R, and A, intertwining the projection maps and preserving the form θ in P,

$$g_P{}^*\theta = \theta,$$

for each $g \in G$, where g_P denotes the diffeomorphism of P induced by g. Similarly, infinitesimal symmetries give rise to vector fields in the space S, Q, V, P, R, and A, related by the projection maps and, for every infinitesimal symmetry ξ, the corresponding vector field ξ_P on P preserves θ,

$$\pounds_{\xi_P}\theta = 0,$$

which is equivalent to

$$\xi_P \lrcorner \, \mathrm{d}\theta = -\mathrm{d}\langle \theta, \xi_P\rangle.$$

The mapping $J_G : P \to g^*$ such that, for every $p \in P$ and every $\xi \in g^*$,

$$\langle J_G(p), \xi\rangle = \langle \theta, \xi_P(p)\rangle,$$

intertwines the action of G in P and the co-adjoint action of G in g^*. Hence, it is a momentum mapping corresponding to the action of G in P. For each $a \in A$, the stability group G_a of a has a Hamiltonian action in the fibre P_a with the momentum map $J_a : P_a \to g_a{}^*$ given by

$$\langle J_a(p), \xi\rangle = \langle \theta, \xi_p(p)\rangle,$$

for every $p \in P_a$, and every $\xi \in g_a$.

Let ξ be an infinitesimal symmetry, and ξ_Z and ξ_V the corresponding vector fields in Z and V, respectively. For each $v \in V$, $\xi_V(v) = \xi_Z v$. Moreover, $\xi_P(p_{PV}(v)) = Tp_{PV}(\xi_V(v))$. Taking into account the expression for the Legendre transformation, we see that $\langle J(p_{PV}(v)), \xi\rangle$ is the value on v of the conserved quantity associated to ξ by the First Noether Theorem.

Let $J_H : P \to h^*$ be a momentum mapping corresponding to the action in P of the gauge group H,

$$\langle J_H(p), \xi\rangle = \langle \theta, \xi_P(p)\rangle,$$

for every $\xi \in h$ and every $p \in P$. The Second Neother Theorem shows that a necessary condition for virtual Cauchy data v V to admit a finite time evolution is the vanishing of $J_H(p_{PV}(v))$. This condition is also sufficient for general relativity and the Yang-Mills theory. For these theories, the final constraint in P is given by the zero level of the momentum map corresponding to the action of the gauge group H,

$$p_{PV}(C) = J_H{}^{-1}(0),$$

where C is the subspace of V consisting of the Cauchy data which admit a finite time evolution.

For the class of theories satisfying all the assumptions made here we have the following result. Let ξ be an infinitesimal symmetry, ξ_P the corresponding vector field on P, hor ξ_P the horizontal component of ξ_P relative to the connection H defined in the preceding section, c_P an integral curve of hor ξ_P such that its projection to S defines a $3 + 1$ splitting of space-time, and s_{YX} the section defined by $c_Q = p_{QP}c_P$.

I. If, for some t_0, $c_P(t_0) \in J_H{}^{-1}(0)$, then $c_P(t) \in J_H{}^{-1}(0)$ for all t, and the section s_{YX} satisfies Euler-Lagrange equations.

II. For each $a \in A$, the restriction to P_a of the vertical component ver ξ_P of ξ_P is the Hamiltonian vector field of $\langle \theta, \xi_P \rangle$ restricted to P_a.

Thus, in order to find solutions of the Euler-Lagrange equations we need to choose an infinitesimal symmetry ξ such that an integral curve c_S of ξ_S defines a $3 + 1$ splitting of space-time, lift c_S to an integral curve c_A of ξ_A and, for each a in $c_A(\mathbf{R})$, find the Hamiltonian vector field in P_a of the restriction of $\langle \theta, \xi_P \rangle$ to P_a. This gives ver ξ_P restricted to $p_{AP}{}^{-1}(c_A(\mathbf{R}))$. The integral curves of hor $\xi_P = \xi_P -$ ver ξ_P, contained in $p_{AP}{}^{-1}(c_A(\mathbf{R}))$ and passing through points in $J_H{}^{-1}(0)$ give rise to solutions s_{YX} of Euler-Lagrange equations. This is an intrinsic version of the adjoint formalism used in [1].

Chapter 3

QUANTIZATION OF GAUGE THEORIES

In the preceding lecture we obtained, for a certain class of field theories, a Hamiltonian formulation of dynamics which is covariant relative to the group G of symmetries of the underlying Lagrangian theory. In this formulation, the evolution takes place in the primary constraint bundle P fibered over the atlas bundle A. The fibration $p_{AP}: P \to A$ is symplectic, and there is an exact 2-form $d\theta$ on P extending the family of symplectic forms along the fibres which defines a connection hor TP. The group G acts in P by structure preserving diffeomorphisms. The mapping $J_G : P \to g^*$ given, for every $p \in P$ and $\xi \in g$, by

$$\langle J_G(p), \xi \rangle = \langle \theta, \xi_P(p) \rangle,$$

intertwines the actions of G in P and in g^*. For each $\xi \in g$, the evolution in the direction ξ_P is given the vector field

$$\text{hor}\, \xi_P = \xi_P - \text{ver}\, \xi_P,$$

where ver ξ_P is the hamiltonian vector field of the momentum

$$J_\xi = \langle \theta, \xi_P \rangle$$

corresponding to ξ; more precisely, for each $a \in A$, the restriction of ver ξ_P to the fibre P_a is the Hamiltonian vector field of $\langle \theta, \xi_P \rangle$ restricted to P_a. Sections s_{YX} of p_{XY} defined by integral curves of hor ξ_P which are contained in the zero level of the momentum map J_H, corresponding to the action in P of the gauge group H, satisfy Euler-Lagrange equations. Conversely, every section s_{YX} satisfying Euler-Lagrange equations gives rise to an integral curve of hor ξ_P contained in $J_H^{-1}(0)$.

8. SELECTION RULES DUE TO GAUGE INVARIANCE

In the formulation of dynamics outlined above the group G of symmetries of the Lagrangian theory and its normal subgroup H of gauge symmetries play fundamental roles. The dynamics is given by the Hamiltonian vector fields of the momentum functions corresponding to elements of the Lie algebra g of G. The constraints are given by the vanishing of the momentum functions corresponding to the elements of the Lie algebra h of the gauge group H. Hence, the first step in a quantization of the system described above should lead to a representation of the symmetry group G. This would give rise to a unitary representation of the gauge group H.

A unitary representation of a locally compact Lie group can be decomposed into its irreducible components. The representation space can be expressed as a direct integral over the space of irreducible unitary representations of the group.

$$\mathcal{H} = \int \mathcal{H}_\rho \, d\mu_\rho ,$$

where $d\mu_\rho$ is a measure on the space of irreducible unitary representations, and $\mathcal{H}_\rho$ is the representation space of a representation ρ. Vectors in $\mathcal{H}$ are given as functions associating to each ρ a vector $\psi(\rho)$ in $\mathcal{H}_\rho$ such that the function $|\psi(\rho)|^2$ is integrable with respect to the measure $d\mu_\rho$. The scalar product of ψ and χ in $\mathcal{H}$ is given by

$$\langle \psi | \chi \rangle = \int \langle \psi(\rho) | \chi(\rho) \rangle \, d\mu_\rho .$$

For each element g of the group, the action of the unitary operator $U(g)$ on ψ in $\mathcal{H}$ is given by

$$(U(g)\psi)(\rho) = U_\rho(g) \, \psi(\rho),$$

where $g \to U(g)$ is the unitary representation ρ of the group in $\mathcal{H}_\rho$.

The gauge group H is not locally compact, and we cannot guarantee that its unitary representation obtained by field quantization decomposes as direct integral of irreducible unitary representations. However, if this were the case, the probability of transition between the states corresponding to different irreducible representations of H would vanish. Moreover, the classical dynamical variables are components of the equivariant mementum mapping J_G. Given ξ in g, we denote by QJ_ξ the quantum operator in $\mathcal{H}$ corresponding to the momentum J_ξ. Since H is a normal subgroup of G it follows that, for each $\eta \in h$,

$$[J_\xi, J_\eta] = J_{[\rho,\eta]} \in h.$$

Therefore, there would exist a family $QJ_{\xi,\rho}$ of operators in $\mathcal{H}_\rho$ such that, for each ψ in $\mathcal{H}$, and each ρ,

$$(QJ_\xi \psi)(\rho) = QJ_{\xi,\rho} \, \psi(\rho).$$

In the situation described above, for each ρ, the evolution of the states transforming under the gauge group H according to the representation ρ would be given by an operator $QJ_{\xi,\rho}$, for an appropriate infinitesimal symmetry ξ, and there would be no interaction between states corresponding to different irreducible representations. If the experimental data indicated that only the states corresponding to some irreducible representations ρ were realized in nature, we could restrict our considerations to the corresponding spaces $\mathcal{H}_\rho$, without violating the invariance of the theory under the group G of symmetries of the underlying classical theory.

All the facts mentioned above are well known from the study of symmetry groups in quantum physics. The gauge groups of field theory differ from symmetry groups in quantum mechanics by their localizability in the space time which leads, via the Second Noether Theorem, to constraints given by the vanishing of the corresponding momentum map. Incorporation into the quantum theory of the classical constraint condition $J_H = 0$ is the next step in quantization of gauge theories; it will be discussed in following section.

9. DIRAC'S QUANTIZATION OF CONSTRAINTS

In 1950 P.A.M. Dirac proposed a quantum implementation of the classical constraint condition $J_H = 0$, requiring that the physically admissible states should be joint eigenstates of the operators QJ_ξ, for all $\xi \in h$, corresponding to the eigenvalue 0. Since the operators QJ_ξ, for $\xi \in h$, generate the action of H in the representation space, Dirac's condition is equivalent to the requirement that the physically admissible states should be gauge invariant. This interpretation of the Dirac's theory of quantization of constraints is supported by the relationship between orbits and representations discussed in Lecture 1.

If 0 is not in the discrete spectrum of the operators QJ_ξ, for $\xi \in h$, then the gauge invariant states will not be normalizable. This is the reason for the contradiction encountered in [4]. In this case the space of physically admissible states consists of generalized eigenvectors of the operators QJ_ξ, defined as follows. Let $\mathcal{H}^\infty$ denote the subspace of $\mathcal{H}$ which is the common domain for the universal enveloping algebra of h. That is $\psi \in \mathcal{H}^\infty$ if and only if, for each $\xi \in h$, and each positive integer n, $(QJ_\xi)^n$ is defined on ψ. The topology in $\mathcal{H}^\infty$ is defined by the system of seminorms $|\psi|_{\xi,n} = |(QJ_\xi)^n \psi|$. Let $\mathcal{H}^{-\infty}$ denote the space of continuous linear functionals on $\mathcal{H}^\infty$. The representation of G in $\mathcal{H}$ estends to a representation in $\mathcal{H}^{-\infty}$; for each $\xi \in g$, and each $\phi \in \mathcal{H}^{-\infty}$, the action of QJ_ξ on ϕ is defined by

$$\langle QJ_\xi \phi, \psi \rangle = \langle \phi, QJ_\xi \psi \rangle$$

for each $\psi \in \mathcal{H}^\infty$. The space of physically admissible states is the subspace of

$H^{-\infty}$ consisting of H invariant elements.

It should be noted that the scalar product in H need not induce a scalar product in the space of physically admissible states, and one will have to determine independently the scalar product in the space of physically admissible. This situation is common in physics. For example, the Klein-Gordon operator is self adjoint in the space of square integrable functions on the space time. However, solutions of the Klein-Gordon equation are not square integrable, and their scalar product involves only integration over spatial variables.

It is possible that the approach to quantization of gauge invariant theories might be helpfull in understanding the phenomenon of confinement of quarks and gluons in quantum chromodynamics and the spontaneous symmetry in the Salam Weinberg model. However, a conclusive test of applicability of this approach to physics of elementary particles would be a construction of quantum chromodynamics along the lines discussed here, including a gauge invariant approximationmethod allowing for quantitative predictions. This is a rahter tall order, and at present we have to rely on indirect evidence provided by a qualitative analysis of the theory, and a quantitative analysis of simplified models.

A convenient finite dimensional model of Yang-Mills theory is provided by $SU(2)$ invariant Yang-Mills fields on the compactification $U(1) \times SU(2)$ of the Minkowski space. In this model the configuration space consists of 3×3 matrices, and the symmetry group is the product of $SO(3)$ with itself, $G = SO(3) \times SO(3)$. For each matrix A and each $g = (L, R) \in G$, $gA = LAR^{-1}$. The Yang-Mills Lagrangian, restricted to $SU(2)$ invariant fields, leads to constrained Hamiltonian dynamics with a Hamiltonian H and the constraints given by the vanishing of the momentum map corresponding to the action of $SO(3)$ given by the left multiplication. Thus, $H = SO(3) \times I$ plays the role of the gauge group, [14].

Schroedinger quantization of this model leads to a unitary representation of G in the space H of square integrable functions $\psi(A)$. This representation can be decomposed into a direct sum of irreducible representations of H. In particular, a wave function $\psi(A)$ belongs to a trivial representation of H if and only if it depends only of $A^T A$. The quantum Hamiltonian QH is given by

$$QH = [(-1/2)\hbar^2 \Delta + V(A)],$$

where the potential energy term V is a quartic polynomial in A tending to infinity as $|A|$ tends to infinity. Hence, QH has a discrete spectrum, and the space of H invariant states is spanned by eigenvectors of QH. Moreover, the ground state of QH is non-degenerate.

It is of interest to enlarge the model by including a Higgs field and study if there is a possibility of a spontaneous symmetry breaking in the resulting quantum theory.

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